Logarithmic stability in determining the time-dependent zero order coefficient in a parabolic equation from a partial Dirichlet-to-Neumann map. Application to the determination of a nonlinear term

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LOGARITHMIC STABILITY IN DETERMINING THE TIME-DEPENDENT ZERO ORDER COEFFICIENT IN A PARABOLIC EQUATION FROM A PARTIAL DIRICHLET-TO-NEUMANN MAP. APPLICATION TO THE DETERMINATION OF A NONLINEAR TERM

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Abstract. We give a new stability estimate for the problem of determining the time-dependent zero order coefficient in a parabolic equation from a partial parabolic Dirichlet-to-Neumann map. The novelty of our result is that, contrary to the previous works, we do not need any measurement on the final time. We also show how this result can be used to establish a stability estimate for the problem of determining the nonlinear term in a semilinear parabolic equation from the corresponding "linearized" Dirichlet-to-Neumann map. The key ingredient in our analysis is a parabolic version of an elliptic Carleman inequality due to Bukhgeim and Uhlmann [6]. This parabolic Carleman inequality enters in an essential way in the construction of CGO solutions that vanish at a part of the lateral boundary.

Keywords: Parabolic equation, Carleman inequality, logarithmic stability, partial Dirichlet-to-Neumann map, semilinear parabolic equation.

Mathematics subject classification: 35R30, 35K20, 35K58.

1. Introduction

Let Ω be a C^2-bounded domain of $\mathbb{R}^n$, $n \geq 2$, with boundary Γ and, for $T > 0$, set

$$Q = \Omega \times (0,T), \quad \Omega_+ = \Omega \times \{0\}, \quad \Sigma = \Gamma \times (0,T).$$

In all of this text, the symbol Δ denotes the Laplace operator with respect to the space variable x.

Consider the initial boundary value problem, abbreviated to IBVP in the sequel,

$$\begin{cases} (\partial_t - \Delta + q(x,t))u = 0 \quad \text{in } Q, \\ u_{|\Omega_+} = 0, \\ u_{|\Sigma} = g. \end{cases} \tag{1.1}$$

We are mainly interested in the stability issue of the problem of determining the time-dependent coefficient q by measuring the corresponding solution u of the IBVP (1.1) on a part of Σ when g is varying in a suitable set of data, which means that we want to establish a stability estimate of recovering q from a partial Dirichlet-to-Neumann map, denoted by DtN map in the sequel.

The IBVP (1.1) is for instance a typical model of the propagation of the heat through a time-evolving homogeneous body. The goal is to determine the coefficient q, who contains some properties of the body, by applying a heat source on some part of the boundary of the body and measuring the temperature on another part of the boundary of the body. Another classical inverse parabolic problem consists in determining the diffusion coefficient of an inhomogeneous medium through an IBVP for the equation $\partial_t v - \mathrm{div}(a(t,x)\nabla v) = 0$. This last problem can be converted to the previous one by means of the Liouville transform $u = \sqrt{a}v$. In many applications we are often lead to determine physical quantities via parabolic IBVP's including nonlinear terms from boundary measurements. For instance such kind of problems appears in reservoir simulation, chemical kinetics and aerodynamics.

We introduce the functional space setting in order to define the DtN map associated to the IBVP (1.1). Following Lions and Magenes [38], $H^{-r,-s}(\Sigma)$, $r,s>0$, denotes the dual space of

$$H^{r,s}_{,0}(\Sigma)=L^2(0,T;H^r(\Gamma))\cap H_0^s(0,T;L^2(\Gamma)).$$

From Proposition 2.3 in Section 2, for $q\in L^\infty(Q)$ and $g\in H^{-\frac12,-\frac14}(\Sigma)$, the IBVP (1.1) admits a unique transposition solution $u_{q,g}\in L^2(Q)$. Additionally, where ν is the unit exterior normal vector field on Γ, the following parabolic DtN map

$$\Lambda_q : H^{-\frac12,-\frac14}(\Sigma)\to H^{-\frac32,-\frac34}(\Sigma)$$
$$g\mapsto \partial_\nu u_{q,g}$$

is bounded.

For $\omega\in\mathbb{S}^{n-1}$, set

$$\Gamma_{\pm,\omega}=\{x\in\Gamma;\ \pm\nu(x)\cdot\omega>0\}$$

and $\Sigma_{\pm,\omega}=\Gamma_{\pm,\omega}\times(0,T)$.

Fix $\omega_0\in\mathbb{S}^{n-1}$, $\mathcal{U}_\pm$ a neighborhood of $\Gamma_{\pm,\omega_0}$ in Γ and set $\mathcal{V}_+=\mathcal{U}_+\times[0,T]$, $\mathcal{V}_-=\mathcal{U}_-\times(0,T)$. Define then the partial parabolic DtN operator

$$\widehat{\Lambda}_q : H^{-\frac12,-\frac14}(\Sigma)\cap\mathscr{E}'(\mathcal{V}_+)\to H^{-\frac32,-\frac34}(\mathcal{V}_-)$$
$$g\mapsto \partial_\nu u_{q,g|\mathcal{V}_-}.$$

Here $\mathscr{E}'(\mathcal{V}_+)=\{u\in\mathscr{E}'(\Gamma\times\mathbb{R});\ \mathrm{supp}(u)\subset\mathcal{V}_+\}$ and $H^{-\frac32,-\frac34}(\mathcal{V}_-)$ denotes the quotient space

$$H^{-\frac32,-\frac34}(\mathcal{V}_-)=\{h=g_{|\mathcal{V}_-};\ g\in H^{-\frac32,-\frac34}(\Sigma)\}.$$

Henceforth, the space $H^{-\frac32,-\frac34}(\mathcal{V}_-)$ is equipped with its natural quotient norm.

We note that $\Lambda_q-\Lambda_{\widetilde q}$ has a better regularity than Λ_q and $\Lambda_{\widetilde q}$ individually. Precisely, Proposition 2.4 in Section 2 shows that actually $\Lambda_q-\Lambda_{\widetilde q}\in\mathscr{B}(H^{-\frac12,-\frac14}(\Sigma),H^{\frac12,\frac14}(\Sigma))$. The same remark is also valid for $\widehat{\Lambda}_q-\widehat{\Lambda}_{\widetilde q}$.

The first author establish in [14] a logarithmic stability estimate for the problem of determining the zero order term from the parabolic DtN map Λ_q together with the final data $g\to u_{q,g}(\cdot,T)$. As a first result in the present work we improve this stability estimate.

In this text, the unit ball of a Banach space X will be denoted in the sequel by B_X.

For $\frac{1}{2(n+3)}<s<\frac{1}{2(n+1)}$, set

$$\Psi_s(\rho)=\rho+|\ln\rho|^{-\frac{1-2s(n+1)}{8}},\quad \rho>0, \tag{1.2}$$

extended by continuity at $\rho=0$ by setting $\Psi_s(0)=0$.

Theorem 1.1. *Fix $m>0$ and $\frac{1}{2(n+3)}<s<\frac{1}{2(n+1)}$. There exists a constant $C>0$, that can depend only on m, Q and s, so that, for any $q,\widetilde q\in mB_{L^\infty(Q)}$,*

$$\|q_1-q_2\|_{H^{-1}(Q)}\leqslant C\Psi_s\left(\|\Lambda_{q_1}-\Lambda_{q_2}\|\right). \tag{1.3}$$

Here $\|\Lambda_{q_1}-\Lambda_{q_2}\|$ stands for the norm of $\Lambda_{q_1}-\Lambda_{q_2}$ in $\mathscr{B}(H^{-\frac12,-\frac14}(\Sigma);H^{\frac12,\frac14}(\Sigma))$.

In the case of the infinite cylindrical domain $Q=\Omega\times(0,\infty)$, Isakov [31] got a stability estimate of determining $q=q(x)$ from the full parabolic DtN map by combining the decay in time of solutions of parabolic equations and the stability estimate in [1] concerning the problem of determining the zero order coefficient in a elliptic BVP from a full DtN map. For finite cylindrical domain $Q=\Omega\times(0,T)$, to our knowledge, even for time-independent coefficients, there is no result in the literature dealing with the stability issue of recovering of q from the full DtN map Λ_q.

In fact Theorem 1.1 is obtained as by-product of the analysis we developed to derive a logarithmic stability estimate for the problem of determining q from the partial parabolic DtN map $\widehat{\Lambda}_q$. This result is stated in the following theorem, where

$$\Phi_s(\rho) = \rho + |\ln|\ln\rho||^{-s}, \quad \rho > 0,\ s > 0, \tag{1.4}$$

extended by continuity at $\rho = 0$ by setting $\Phi_s(0) = 0$.

Theorem 1.2. *Let $m > 0$, there exist two constants $C > 0$ and $s \in (0, 1/2)$, that can depend only on m, Q and $\mathcal{V}_\pm$, so that, for any $q, \widetilde{q} \in mB_{L^\infty(Q)}$,*

$$\|q_1 - q_2\|_{H^{-1}(Q)} \leqslant C\Phi_s\left(\|\widehat{\Lambda}_q - \widehat{\Lambda}_{\widetilde{q}}\|\right). \tag{1.5}$$

Here $\|\widehat{\Lambda}_q - \widehat{\Lambda}_{\widetilde{q}}\|$ denotes the norm of $\widehat{\Lambda}_q - \widehat{\Lambda}_{\widetilde{q}}$ in $\mathscr{B}(H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma); H^{\frac{1}{2},\frac{1}{4}}(\mathcal{V}_-))$.

It is worth mentioning that the uniqueness holds for the problem of determining q from the partial DtN operator that maps the boundary condition g supported on $\Gamma_0 \times (0,T)$ into $\partial_\nu u_{q,g}$ restricted to $\Gamma_1 \times (0,T)$, where Γ_i, $i = 0, 1$, are arbitrary nonempty open subsets of Γ. This result is stated as Theorem 3.27 in [14, page 197]. We note that the stability estimate corresponding to this uniqueness result remains an open problem.

In order to avoid the data at the final time, we adopt a strategy based on a parabolic Carleman inequality to construct the so-called CGO solutions vanishing at a part of the lateral boundary similar to that already used by the second author in [33, 34, 35] for determining time-dependent coefficients in a wave equation.

There is a wide literature devoted to inverse parabolic problems and specifically the determination of time-dependent coefficients. We just present briefly some typical results. Canon and Esteva [7] proved a logarithmic stability estimate for the determination of the support of a source term in a one dimension parabolic equation from a boundary measurement. This result was extended to three dimension heat equation in [8]. The case of a non local measurement was considered by Canon and Lin in [9, 10]. In [12], the first author proved existence, uniqueness and Lipschitz stability for the determination of a time-dependent coefficient appearing in an abstract integro-differential equation, extending earlier results in [13]. The first author and Yamamoto established in [22] a stability estimate for the inverse problem of determining a source term appearing in a heat equation from Neumann boundary measurements. In [23], the first author and Yamamoto considered an inverse semi-linear parabolic problem of recovering the coefficient used to reach a desired temperature along a curve. In [28], Isakov extended the construction of complex geometric optics solutions, introduced in [40], to various PDE's including hyperbolic and parabolic equations to prove the density of products of solutions. One can get from the results in [28] the unique determination of q from the measurements on the lateral boundary together with data at the final time. When the space domain is cylindrical, adopting the strategy introduced in [5], the second author and Gaïtan proved in [25] that the time-dependent zero order coefficient can be recovered uniquely from a single boundary measurement. Based on properties of fundamental solutions of parabolic equations, we proved in [15] Lipschitz stability of determining the time-dependent part of the zero order coefficient in a parabolic IBVP from a single boundary measurement.

We also mention the recent works related to the determination of a time-dependent coefficients in IBVP's for hyperbolic, fractional diffusion and dynamical Schrödinger equations [3, 16, 18, 24, 33, 34, 35].

We point out that, concerning the elliptic case, a stability estimate corresponding to the uniqueness result by Bukhgeim and Uhlmann [6] was established by Heck and Wang [27]. This result was improved by the authors and Soccorsi in [17]. Caro, Dos Santos Ferreira and Ruiz [11] obtained recently a logarithmic stability estimate corresponding to the uniqueness result by Kenig, Sjöstrand and G. Uhlmann [32]. Both the determination of the scalar potential and the conductivity in a periodic cylindrical domain from a partial DtN map was tackled in [19, 20]. We just quote these few references. But, of course, there is a tremendous literature on this subject in connection with the famous Calderòn's problem.

Considering time-dependent unknown coefficients in parabolic equations is very useful when treating the determination of the nonlinear term appearing in a semilinear parabolic equation. We discuss this topic in Section 6. Uniqueness results for such kind of inverse semilinear parabolic problems was already established by Isakov [29, 30, 31]. Stability estimates and uniqueness in the case of a single boundary lateral measurement has been proved in [21, 36] for a restricted class of unknown nonlinearities.

The rest of this text is organized as follows. Section 2 is devoted to existence and uniqueness of solutions of the IBVP (1.1) in a weak sense. Following a well established terminology, we call these weak solutions the transposition solutions. We prove in Section 3 a Carleman inequality which is, as we said before, the key point in constructing CGO solutions that vanish on a part of the lateral boundary. These CGO solutions are constructed in Section 4. Theorems 1.1 and 1.2 are proved in Section 5. We finally apply in section 6 the result in Theorem 1.1 to the problem of determining the non linear term is a semilinear parabolic equation from the corresponding "linearized" DtN map.

2. Transposition solutions

This section is mainly dedicated to the IBVP (1.1). We construct the necessary framework leading to the rigorous definition of the parabolic DtN map.

For sake of simplicity, we limit our study in this section to real-valued functions. But all the results are extended without any difficulty to complex-valued functions.

Henceforth

$$H_\pm = \{u \in L^2(Q);\ (\pm\partial_t - \Delta)u \in L^2(Q)\}$$

is equiped with its natural norm

$$\|u\|_{H_\pm} = \left(\|u\|^2_{L^2(Q)} + \|(\pm\partial_t - \Delta)u\|^2_{L^2(Q)}\right)^{1/2}.$$

Proceeding similarly to [34, Theorem 4] or [19, Lemma 2.1], we show that $\mathcal{C}^\infty(\overline{Q})$ is dense in $H_\pm$.

In the rest of this text, $\Omega_- = \Omega \times \{T\}$.

Denote by $\mathcal{N}$ the set of $(g_0, g_1, w_+, w_-) \in H^{\frac{3}{2},\frac{3}{4}}(\Sigma) \oplus H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \oplus H^1(\Omega) \oplus H^1(\Omega)$ satisfying the compatibility conditions

$$g_0(\cdot, 0) = w_+ \ \text{ and } \ g_0(\cdot, T) = w_- \ \text{ on } \Gamma. \tag{2.1}$$

From the results in [38, Section 2.5, page 17], for any $(g_0, g_1, w_+, w_-) \in \mathcal{N}$, there exists

$$w = E(g_0, g_1, w_+, w_-) \in H^{2,1}(Q)$$

such that, in the trace sense,

$$w_{|\Sigma} = g_0, \quad \partial_\nu w_{|\Sigma} = g_1, \quad w_{|\Omega_\pm} = w_\pm \tag{2.2}$$

and

$$\|w\|_{H^{2,1}(Q)} \leqslant C\left(\|g_0\|_{H^{\frac{3}{2},\frac{3}{4}}(\Sigma)} + \|g_1\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} + \|w_-\|_{H^1(\Omega)} + \|w_+\|_{H^1(\Omega)}\right), \tag{2.3}$$

for some constant $C > 0$ depending only on Q.

We recall that $H_0^{\frac{1}{4}}(0, T; L^2(\Gamma))$ coincides with $H^{\frac{1}{4}}(0, T; L^2(\Gamma))$. This fact is more or less known, but for sake of completeness we provide its proof in Lemma A.1 of Appendix A. Whence $H^{\frac{1}{2},\frac{1}{4}}_{,0}(\Sigma)$ is identified to $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$. Therefore, we identify in the sequel the dual space of $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ to $H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$.

Proposition 2.1. *The maps τ_j, $j = 0, 1$, and $r_\pm$ defined for $v \in \mathcal{C}^\infty(\overline{Q})$ by*

$$\tau_0 v = v_{|\Sigma}, \quad \tau_1 v = \partial_\nu v_{|\Sigma}, \quad r_\pm v = v_{|\Omega_\pm} \tag{2.4}$$

are extended to bounded operators

$$\tau_0 : H_\pm \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma), \quad \tau_1 : H_\pm \to H^{-\frac{3}{2},-\frac{3}{4}}(\Sigma), \quad r_-, r_+ : H_\pm \to H^{-1}(\Omega). \tag{2.5}$$

Proof. Assume that $w = E(0, g_1, 0, 0)$ for some $(0, g_1, 0, 0) \in \mathcal{N}$. Then Green's formula, where $v \in \mathcal{C}^\infty(\overline{Q})$, yields

$$\langle \tau_0 v, \tau_1 w\rangle = \int_Q (\partial_t - \Delta) v w dxdt - \int_Q v(-\partial_t - \Delta) w dxdt$$

which, combined with (2.2) and (2.3), entails

$$|\langle \tau_0 v, g_1\rangle| \leqslant C \|v\|_{H_+} \|w\|_{H^{2,1}(Q)} \leqslant C \|v\|_{H_+} \|g_1\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)}.$$

Whence τ_0 can be extended by density to a bounded operator from H_+ to $H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$.

We have similarly, by taking $w = E(g_0, 0, 0, 0)$ for some $(g_0, 0, 0, 0) \in \mathcal{N}$,

$$-\langle \tau_1 v, g_0\rangle = \int_Q (\partial_t - \Delta) v w dxdt - \int_Q v(-\partial_t - \Delta) w dxdt.$$

This formula, (2.2) and (2.3) imply

$$|\langle \tau_1 v, g_0\rangle| \leqslant C \|v\|_{H_+} \|w\|_{H^{2,1}(Q)} \leqslant C \|v\|_{H_+} \|g_0\|_{H_{,0}^{\frac{3}{2},\frac{3}{4}}(\Sigma)}.$$

Consequently, τ_1 can be extended by density to a bounded operator from H_+ to $H^{-\frac{3}{2},-\frac{3}{4}}(\Sigma)$.

The remaining part of the proof can be proved in a similar manner. We leave to the interested reader to write down the details. ☐

Introduce

$$\mathcal{H}_\pm = \{\tau_0 u;\ u \in H_+ \text{ and } r_\pm u = 0\},$$
$$S_\pm = \{u \in L^2(Q);\ (\pm\partial_t - \Delta)u = 0 \ \text{ and } r_\pm u = 0\}.$$

Proposition 2.2. *We have $\tau_0 S_\pm = \mathcal{H}_\pm = H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ and, for any $u \in S_+$,*

$$\|u\|_{H_+} = \|u\|_{L^2(Q)} \leqslant C \|\tau_0 u\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}, \tag{2.6}$$

for some constant $C > 0$ depending only on Q.

Proof. We already know that $\tau_0 S_\pm \subset \mathcal{H}_\pm \subset H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$. Then we have only to prove the reverse inclusions. To do that, fix $g \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ and, for $F \in L^2(Q)$, consider the IBVP

$$\begin{cases} (-\partial_t - \Delta)w = F \ \text{ in } Q, \\ w_{|\Omega_- \cup \Sigma} = 0. \end{cases}$$

According to [14, Theorem 1.43], this IBVP admits a unique solution $w = w_F \in H^{2,1}(Q)$ so that

$$\|\tau_1 w_F\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \leqslant C \|w_F\|_{H^{2,1}(Q)} \leqslant C \|F\|_{L^2(Q)}. \tag{2.7}$$

Thus, the linear form on $L^2(Q)$ given by

$$F \in L^2(Q) \mapsto -\langle g, \tau_1 w_F\rangle$$

is continuous. Therefore, according to Riesz's representation theorem, there exists $u \in L^2(Q)$ such that

$$(u, F) = -\langle g, \tau_1 w_F\rangle.$$

Here and henceforth $(\cdot,\cdot)$ is the usual scalar product on $L^2(Q)$.

Taking $F = (-\partial_t - \Delta)w$ for some $w \in \mathcal{C}_0^\infty(Q)$, we get $(\partial_t - \Delta)u = 0$ on Q. On the other hand, the choice of $F = (-\partial_t - \Delta)w$, where $w = E(0, 0, w_+, 0) \in H^{2,1}(Q)$ with $(0, 0, w_+, 0) \in \mathcal{N}$, yields $r_+ u = 0$ and then $u \in S_+$. Finally, $F = (-\partial_t - \Delta)w$, for some $w = E(0, g_1, 0, 0) \in H^{2,1}(Q)$ with $(0, g_1, 0, 0) \in \mathcal{N}$, gives $\tau_0 u = g$. In other words, we proved that $g \in \tau_0 S_+$ and consequently $\tau_0 S_+ = \mathcal{H}_\pm = H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$. We complete the proof by noting that (2.6) follows from (2.7) and the analysis we carried out for S_+ can be adapted with slight modifications to S_-. ☐

For $\varepsilon = \pm$, consider the IBVP

$$\begin{cases} (\varepsilon\partial_t - \Delta + q(x,t))u = 0 \quad \text{in } Q, \\ u_{|\Omega_\varepsilon} = 0, \\ u_{|\Sigma} = g. \end{cases} \tag{2.8}$$

Proposition 2.3. *For $g \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ and $q \in mB_{L^\infty(Q)}$, the IBVP (2.8) admits a unique transposition solution $u_{q,g}^\varepsilon \in H_\varepsilon$ satisfying*

$$\left\| u_{q,g}^\varepsilon \right\|_{H_\varepsilon} \leqslant C \left\| g \right\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}, \tag{2.9}$$

where the constant C depends only on Q and m. Additionally the parabolic DtN map

$$\Lambda_q : g \mapsto \tau_1 u_{q,g}^+$$

defines a bounded operator from $H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ into $H^{-\frac{3}{2},-\frac{3}{4}}(\Sigma)$.

Proof. We give the proof for $\varepsilon = +$. The case $\varepsilon = -$ can be treated similarly. By Proposition 2.2, there exists a unique $G \in S_+$ such that $\tau_0 G = g$ and

$$\|G\|_{L^2(Q)} \leqslant C \|g\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}.$$

Consider the IBVP

$$\begin{cases} (\partial_t - \Delta + q)w = -qG \quad \text{in } Q, \\ w_{|\Omega_+ \cup \Sigma} = 0. \end{cases}$$

As $-qG \in L^2(Q)$, we get from [14, Theorem 1.43] that this IBVP has a unique solution $w \in H^{2,1}(Q)$ satisfying

$$\|w\|_{H^{2,1}(Q)} \leqslant C \|qG\|_{L^2(Q)} \leqslant C \|q\|_{L^\infty(Q)} \|g\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}. \tag{2.10}$$

Hence, $u_{q,g}^+ = w + G \in H_+$ is the unique transposition solution of (2.8) and (2.9) follows from (2.10).

Now, according to Proposition 2.1, we have $\tau_1 u_{q,g}^+ \in H^{-\frac{3}{2},-\frac{3}{4}}(\Sigma)$ with

$$\begin{aligned} \left\|\tau_1 u_{q,g}^+\right\|_{H^{-\frac{3}{2},-\frac{3}{4}}(\Sigma)}^2 \leqslant C \left\|u_{q,g}^+\right\|_{H_+}^2 &= C \left(\left\|u_{q,g}^+\right\|_{L^2(Q)}^2 + \left\|q u_{q,g}^+\right\|_{L^2(Q)}^2 \right) \\ &\leqslant C \left(1 + \|q\|_{L^\infty(Q)}^2\right) \left\|u_{q,g}^+\right\|_{L^2(Q)}^2, \end{aligned}$$

which, in combination with (2.9), entails

$$\Lambda_q : H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \to H^{-\frac{3}{2},-\frac{3}{4}}(\Sigma) : g \mapsto \tau_1 u_{q,g}^+$$

defines a bounded operator. $\square$

The identity in the following proposition will be very useful in our analysis.

Proposition 2.4. *Fix $m > 0$ and let $q, \widetilde{q} \in mB_{L^\infty(Q)}$. Then $\Lambda_q - \Lambda_{\widetilde{q}}$ is a bounded operator from $H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ into $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ and*

$$\langle (\Lambda_q - \Lambda_{\widetilde{q}})g, h \rangle = \int_Q (q - \widetilde{q}) u_{q,g}^+ u_{\widetilde{q},h}^- dxdt, \quad g, h \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma). \tag{2.11}$$

Proof. It is straightforward to check that $u = u_{\widetilde{q},g}^+ - u_{q,g}^+$ is the solution of the IBVP

$$\begin{cases} (\partial_t - \Delta + \widetilde{q})u = (q - \widetilde{q}) u_{q,g}^+ \ \text{ in } Q, \\ u_{|\Omega_+ \cup \Sigma} = 0. \end{cases}$$

Therefore, $u \in H^{2,1}(Q)$ and by the trace theorem [38, Theorem 2.1, page 9] $(\Lambda_q - \Lambda_{\widetilde{q}})g = \partial_\nu u \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$. Additionally (2.9) implies

$$\begin{aligned}\|(\Lambda_q - \Lambda_{\widetilde{q}})g\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} = \|\partial_\nu u\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} &\leqslant C\,\|u\|_{H^{2,1}(Q)} \\ &\leqslant C\left\|(q-\widetilde{q})u^+_{q,g}\right\|_{L^2(Q)} \\ &\leqslant C\,\|g\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)},\end{aligned}$$

where C is a generic constant depending only on Q and m.

Let $h = \tau_0 H$, where $H \in \mathcal{C}^\infty(\overline{Q}) \cap S_-$. We find by making integrations by parts

$$\begin{aligned}-\int_\Sigma \partial_\nu u h\, d\sigma dt &= \int_Q (\partial_t - \Delta + \widetilde{q})u u^-_{\widetilde{q},h} dxdt - \int_Q u(-\partial_t - \Delta + \widetilde{q})u^-_{\widetilde{q},h} dxdt \\ &= \int_Q (q-\widetilde{q})u^+_{q,g} u^-_{\widetilde{q},h} dxdt.\end{aligned} \tag{2.12}$$

As $\mathscr{S}_- = C^\infty(\overline{Q}) \cap S_-$ is dense in S_-, $\tau_0\mathscr{S}_-$ is dense in $H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ according to Proposition 2.2. Whence (2.11) is deduced from (2.12) by density. □

3. A Carleman inequalities

We establish in this section a parabolic version of an elliptic Carleman inequality due to Bukgheim and Uhlmann [6]. This inequality is used in an essential way in constructing CGO solutions vanishing at a part of the lateral boundary.

In this section $\varepsilon = \pm$ and, for $\omega \in \mathbb{S}^{n-1}$ and $\rho \geq 0$,

$$\varphi_{\varepsilon,\omega,\rho}(x,t) = e^{-\varepsilon 2(\rho\omega\cdot x + \rho^2 t)}, \ \ (x,t) \in \overline{Q},$$

and $\psi_{\varepsilon,\omega,\rho} = \sqrt{\varphi_{\varepsilon,\omega,\rho}}$.

Observe that $\psi_{\varepsilon,\omega,\rho}$ satisfies

$$(\varepsilon\partial_t - \Delta)\psi_{\varepsilon,\omega,\rho} = 0, \ \ (x,t) \in \overline{Q}.$$

Theorem 3.1. *There exists a constant $C > 0$ depending only on Q with the property that, for any $m > 0$, we find $\rho_0 > 0$, depending only on Q and m so that, for any $q \in mB_{L^\infty(Q)}$, $\rho \geq \rho_0$ and $u \in C^2(\overline{Q})$ satisfying $u = 0$ on $\Sigma \cup \Omega_\varepsilon$,*

$$\begin{aligned}&\int_{\Omega_{-\varepsilon}} \varphi_{\varepsilon,\omega,\rho}\,|u|^2\,dx + \rho\int_{\Sigma_{\varepsilon,\omega}} \varphi_{\varepsilon,\omega,\rho}\,|\partial_\nu u|^2\,|\omega\cdot\nu|\,d\sigma dt + \rho^2\int_Q \varphi_{\varepsilon,\omega,\rho}\,|u|^2\,dxdt \\ &\qquad\leq C\left(\int_Q \varphi_{\varepsilon,\omega,\rho}\,|(\varepsilon\partial_t - \Delta + q)u|^2\,dxdt + \rho\int_{\Sigma_{-\varepsilon,\omega}} \varphi_{\varepsilon,\omega,\rho}\,|\partial_\nu u|^2\,|\omega\cdot\nu|\,d\sigma dt\right).\end{aligned} \tag{3.1}$$

Proof. It is enough to give the proof in the case of real-valued functions. We consider the case $\varepsilon = +$. The case $\varepsilon = -$ can be treated similarly. Let $u \in C^2(\overline{Q})$ satisfying $u = 0$ on $\Sigma \cup \Omega_+$ and set $v = \psi_{+,\omega,\rho}u$. Straightforward computations give

$$\psi_{+,\omega,\rho}(\partial_t - \Delta)u = P_{\omega,\rho}v, \tag{3.2}$$

where

$$P_{\omega,\rho} = \partial_t - \Delta - 2\rho\omega\cdot\nabla.$$

In this proof, the symbol ∇ denotes the gradient with respect to the variable x.

We split $P_{\omega,\rho}$ into two terms

$$P_{\omega,\rho} = -\Delta + Q_{\omega,\rho} \ \text{ with } \ Q_{\omega,\rho} = \partial_t - 2\rho\omega\cdot\nabla.$$

Hence
$$\|P_{+,\rho,\omega}v\|_{L^2(Q)}^2 \geqslant \|Q_{\omega,\rho}v\|^2 - 2\left(\Delta v, Q_{\omega,\rho}v\right).$$
But
$$-2\left(\Delta v, Q_{\omega,\rho}v\right)_{L^2(Q)} = -2\int_Q \Delta v\partial_t v dxdt + 2\rho\int_Q \Delta v\omega\cdot\nabla v dxdt.$$
Applying Green's formula, we get
$$-2\int_Q \Delta v\partial_t v dxdt = \int_Q \partial_t|\nabla v|^2 dxdt = \int_{\Omega_-}|\nabla v|^2 dx.$$
On the other hand, we have from the proof of [6, Lemma 2.1]
$$2\int_Q \Delta v\omega\cdot\nabla v dxdt = \int_\Sigma |\partial_\nu v|^2\omega\cdot\nu d\sigma dt.$$
Combining these formulas, we obtain
$$\|P_{\omega,\rho}v\|_{L^2(Q)}^2 \geqslant \int_{\Omega_-}|\nabla_x v|^2 + \rho\int_\Sigma |\partial_\nu v|^2\omega\cdot\nu d\sigma dt + \|Q_{\omega,\rho}v\|_{L^2(Q)}^2. \tag{3.3}$$

We need the following Poincaré type inequality to pursue the proof. This inequality is proved later in this text.

Lemma 3.1. *There exists a constant C, that can depend only on Ω so that, for any $\rho > 2$ and $v \in H^1(Q)$ satisfying $v = 0$ on $\Sigma\cup\Omega_\varepsilon$,*
$$\rho\,\|v\|_{L^2(Q)} \leqslant C\,\|Q_{\omega,\rho}v\|_{L^2(Q)}. \tag{3.4}$$

Inequality (3.4) in (3.3) yields
$$\int_{\Omega_-}|v|^2dx + \rho\int_{\Sigma_+}|\partial_\nu v|^2\omega\cdot\nu d\sigma dt + \rho^2\,\|v\|_{L^2(Q)}^2 \leqslant C_0\left(\|P_{\omega,\rho}v\|_{L^2(Q)}^2 + \rho\int_{\Sigma_-}|\partial_\nu v|^2|\omega\cdot\nu|d\sigma dt\right).$$
Here the constant C_0 depends only on Q. This gives (3.1) when $q = 0$. For an arbitrary $q \in mB_{L^\infty(Q)}$, we have
$$|\partial_t u - \Delta u|^2 = |\partial_t u - \Delta u + qu - qu|^2 \leq 2\left|(\partial_t - \Delta + q)u\right|^2 + 2m^2\left|u\right|^2.$$
Fix $\rho_0 > 2\max(\sqrt{C_0}m, 1)$. Then (3.1) follows with $C = 2C_0$. □

Proof of Lemma 3.1. We consider the case $\varepsilon = +$. The case $\varepsilon = -$ is proved similarly.

Let $v \in H^1(Q)$ satisfying $v = 0$ on $\Sigma\cup\Omega_+$. A classical reflexion argument in t with respect to T shows that v is the restriction to Q of a function belonging to $H_0^1(\Omega\times(0,2T))$. Therefore, by density, it is enough to give the proof when $v \in C_0^\infty(\Omega\times(0,2T))$ that we consider in the sequel as a subset of $C_0^\infty(\Omega\times(0,+\infty))$.

Let $v \in C_0^\infty(\Omega\times(0,+\infty))$. If $\eta = (\eta_x,\eta_t) \in \mathbb{R}^n\times\mathbb{R}$ is defined by
$$\eta_x = \frac{-2\rho\omega}{\sqrt{1+4\rho^2}},\quad \eta_t = \frac{1}{\sqrt{1+4\rho^2}},$$
then
$$v(x,t) = \int_{-\infty}^0 \partial_s v((x,t)+s\eta)ds,\quad (x,t)\in Q.$$
Fix $R > 0$ such that $\Omega \subset \{|x| \leqslant R;\ x\in\mathbb{R}^n\}$. For $s < -4R$, $x \in \Omega$ and $\rho > 2$, we get
$$|x + s\eta_x| \geqslant \frac{-s}{2} - |x| > R.$$
Thus, for $s < -4R$ and $x\in\Omega$, $x + s\eta_x \notin \Omega$ yielding $v((x,t)+s\eta) = 0$. Therefore
$$|v(x,t)|^2 \leqslant \left|\int_{-4R}^0 \partial_s v((x,t)+s\eta)ds\right|^2,\quad (x,t)\in Q.$$

Applying Cauchy-Schwarz's inequality, we obtain

$$\int_Q |v(x,t)|^2 dxdt \le 4R \int_Q \int_{-4R}^0 |\partial_s v((x,t)+s\eta)|^2 dsdxdt$$
$$= \frac{4R}{1+4\rho^2}\int_Q \int_{-4R}^0 |Q_{\omega,\rho} v((x,t)+s\eta)|^2 dsdxdt.$$

We make the substitution $\tau = t + s\eta_t$, $y = x + s\eta_x$ and we apply Fubini's theorem. We find

$$\int_Q |v(x,t)|^2 dxdt \leqslant \frac{4R}{1+4\rho^2}\int_{-4R}^0 \int_{-\infty}^{T+s\eta_t} \int_{\mathbb{R}^n} |Q_{\omega,\rho} v(y,\tau)|^2 dyd\tau ds$$
$$\leqslant \frac{16R^2}{1+4\rho^2}\int_Q |Q_{\omega,\rho} v(x,t)|^2 dxdt.$$

Here we used both the fact that $\eta_t > 0$ and $\mathrm{supp}(v) \subset (0,+\infty)\times\Omega$. □

4. CGO solutions vanishing on a part of the lateral boundary

With the help of the Carleman inequality in the previous section, we construct in this section CGO solutions vanishing at a part of the lateral boundary.

As in the preceding section, for $\omega \in \mathbb{S}^{n-1}$, $\varepsilon = \pm$ and $\rho \ge 0$,

$$\varphi_{\varepsilon,\omega,\rho}(x,t) = e^{-\varepsilon 2(\rho\omega\cdot x + \rho^2 t)}, \quad (x,t) \in \overline{Q},$$

and $\psi_{\varepsilon,\omega,\rho} = \sqrt{\varphi_{\varepsilon,\omega,\rho}}$. Set, where $\delta > 0$ and $\omega \in \mathbb{S}^{n-1}$,

$$\Gamma_{+,\omega,\delta} = \{x \in \Gamma;\ \nu(x)\cdot\omega > \delta\}$$

and $\Sigma_{+,\omega,\delta} = \Gamma_{+,\omega,\delta} \times (0,T)$.

Fix $\omega \in \mathbb{S}^{n-1}$, $\xi \in \mathbb{R}^n$ with $\xi\cdot\omega = 0$, $\tau \in \mathbb{R}$ and $\rho \ge \rho_0$, ρ_0 is as in Theorem 3.1. Let $\zeta = (\xi,\tau)$ and

$$\theta_+(x,t) = \left(1 - e^{-\rho^{\frac{3}{4}}t}\right) e^{-i(x,t)\cdot\zeta}$$
$$\theta_-(x,t) = \left(1 - e^{-\rho^{\frac{3}{4}}(T-t)}\right).$$

Theorem 4.1. *Let $m > 0$. There exists a constant $C > 0$ depending only on Q, m and δ so that, for any $q \in mB_{L^\infty(\Omega)}$, the equation*

$$(\pm\partial_t - \Delta + q)u = 0 \ \text{ in } Q$$

has a solution $u_{\pm,q} \in \mathcal{H}_\pm$, satisfying $u_{\pm,q} = 0$ on $\Sigma_{+,\mp\omega,\delta}$, of the form

$$u_{\pm,q} = \psi_{\mp,\omega,\rho}\left(\theta_\pm + w_{\pm,q}\right), \tag{4.1}$$

where $w_{\pm,q} \in H_\pm$ is such that

$$\|w_{+,q}\|_{L^2(Q)} \leqslant C(\rho^{-\frac{1}{4}} + \rho^{-1}\langle\zeta\rangle^2), \quad \|w_{-,q}\|_{L^2(Q)} \leqslant C\rho^{-\frac{1}{4}}. \tag{4.2}$$

Here and henceforth $\langle\zeta\rangle = \sqrt{1+|\zeta|^2}$.

In the rest of this section, for sake of simplicity, we use φ_ε and ψ_ε instead of $\varphi_{\varepsilon,\omega,\rho}$ and $\psi_{\varepsilon,\omega,\rho}$.

Before proving Theorem 4.1, we establish some preliminary results. We firstly rewrite the Carleman inequality in Theorem 3.1 as an energy inequality in weighted L^2-spaces. To this end, denote by $\|\cdot\|_{Q,\varphi_\varepsilon}$, $\|\cdot\|_{\Omega_{-\varepsilon},\varphi_\varepsilon}$ and $\|\cdot\|_{\Sigma_{\pm\varepsilon},\varphi_\varepsilon\gamma}$, where $\gamma = |\nu\cdot\omega|$, the respective L^2-norms of $L^2(Q,\varphi_\varepsilon dxdt)$, $L^2(\Omega_{-\varepsilon},\varphi_\varepsilon dx)$ and $L^2(\Sigma_{\pm\varepsilon,\omega},\varphi_\varepsilon\gamma d\sigma dt)$. Under these notations, inequality (3.1) takes the form

$$\|u\|_{\Omega_{-\varepsilon},\varphi_\varepsilon} + \rho^{\frac{1}{2}}\|\partial_\nu u\|_{\Sigma_\varepsilon,\varphi_\varepsilon\gamma} + \rho\|u\|_{Q,\varphi_\varepsilon} \leqslant C\left(\|(\varepsilon\partial_t - \Delta + q)u\|_{Q,\varphi_\varepsilon} + \|\partial_\nu u\|_{\Sigma_{-\varepsilon},\varphi_\varepsilon\gamma}\right), \tag{4.3}$$

for any $\rho \geq \rho_0$ and $u \in \mathcal{D}_\varepsilon = \{v \in C^2(\overline{Q});\ \ v_{|\Sigma\cup\Omega_\varepsilon} = 0\}$.

Again, for sake of simplicity, we use in the sequel the following notations

$$L^2_{Q,\varphi_\varepsilon} = L^2(Q, \varphi_\varepsilon dxdt) \quad L^2_{\Omega_{-\varepsilon},\varphi_\varepsilon} = L^2(\Omega_{-\varepsilon}, \varphi_\varepsilon dx), \quad L^2_{\Sigma_{\pm\varepsilon},\varphi_\varepsilon\gamma} = L^2(\Sigma_{\pm\varepsilon,\omega}, \varphi_\varepsilon \gamma d\sigma dt).$$

We identify the dual space of $L^2_{Q,\varphi_\varepsilon}$ (resp. $L^2_{\Sigma_{-\varepsilon},\varphi_\varepsilon\gamma}$) by $L^2_{Q,\varphi_\varepsilon^{-1}}$ (resp. $L^2_{\Sigma_{-\varepsilon},\varphi_\varepsilon^{-1}\gamma^{-1}}$). The respective norms of $L^2_{Q,\varphi_\varepsilon^{-1}}$ and $L^2_{\Sigma_{-\varepsilon},\varphi_\varepsilon^{-1}\gamma^{-1}}$ are denoted respectively by $\|\cdot\|_{Q,\varphi_\varepsilon^{-1}}$ and $\|\cdot\|_{\Sigma_{-\varepsilon},\varphi_\varepsilon^{-1}\gamma^{-1}}$.

Consider the following subspace of $L^2_{Q,\varphi_\varepsilon} \oplus L^2_{\Sigma_{-\varepsilon},\varphi_\varepsilon\gamma}$

$$\mathcal{M}_\varepsilon = \{((\varepsilon\partial_t - \Delta + q)v, \partial_\nu v_{|\Sigma_{-\varepsilon,\omega}});\ v \in \mathcal{D}_\varepsilon\}.$$

Lemma 4.1. *Fix $m > 0$ and $q \in mB_{L^\infty(Q)}$. Assume that $\rho \geqslant \rho_0$, with ρ_0 the constant of Theorem 3.1, and let $(G, h) \in L^2_{Q,\varphi_\varepsilon^{-1}} \oplus L^2_{\Sigma_\varepsilon,\varphi_\varepsilon^{-1}\gamma^{-1}}$. Then there exists $z \in L^2_{Q,\varphi_\varepsilon}$ satisfying*

$$\begin{aligned}
&(-\varepsilon\partial_t - \Delta + q)z = G \ \ \text{in } Q,\\
&z_{|\Sigma_{\varepsilon,\omega}} = h,\\
&z_{|\Omega_{-\varepsilon}} = 0,\\
&\|z\|_{Q,\varphi_\varepsilon} \leqslant C\left(\rho^{-1}\|G\|_{Q,\varphi_\varepsilon^{-1}} + \rho^{-\frac{1}{2}}\|h\|_{\Sigma_{-\varepsilon},\varphi_\varepsilon^{-1}\gamma^{-1}}\right),
\end{aligned}$$

for some constant $C > 0$ depending only on Q and m.

Proof. Define on $\mathcal{M}_\varepsilon$ the map $\mathscr{S}$ by

$$\mathscr{S}[((\varepsilon\partial_t - \Delta + q)f, \partial_\nu f_{|\Sigma_{-\varepsilon,\omega}})] = \langle f, G\rangle_{L^2(Q)} - \langle \partial_\nu f, h\rangle_{L^2(\Sigma_{\varepsilon,\omega})}, \quad f \in \mathcal{D}_\varepsilon.$$

In light of (4.3) we get, for $f \in \mathcal{D}_\varepsilon$,

$$\begin{aligned}
|\mathscr{S}[((\varepsilon\partial_t - \Delta + q)f, \partial_\nu f_{|\Sigma_{\varepsilon,\omega}})]| &\leq \|f\|_{Q,\varphi_\varepsilon}\|G\|_{Q,\varphi_\varepsilon^{-1}} + \|\partial_\nu f\|_{\Sigma_\varepsilon,\varphi_\varepsilon\gamma}\|h\|_{\Sigma_\varepsilon,\varphi_\varepsilon^{-1}\gamma^{-1}}\\
&\leq \left(\rho^{-1}\|G\|_{Q,\varphi_\varepsilon^{-1}} + \rho^{-\frac{1}{2}}\|h\|_{\Sigma_\varepsilon,\varphi_\varepsilon^{-1}\gamma^{-1}}\right)(\rho\|f\|_{Q,\varphi_\varepsilon} + \|\partial_\nu f\|_{\Sigma_\varepsilon,\varphi_\varepsilon\gamma})\\
&\leq C\left(\rho^{-1}\|G\|_{Q,\varphi_\varepsilon^{-1}} + \rho^{-\frac{1}{2}}\|h\|_{\Sigma_\varepsilon,\varphi_\varepsilon^{-1}\gamma^{-1}}\right)\left(\|(\varepsilon\partial_t - \Delta + q)u\|_{Q,\varphi_\varepsilon} + \|\partial_\nu u\|_{\Sigma_{-\varepsilon},\varphi_\varepsilon\gamma}\right)\\
&\leq C\left(\rho^{-1}\|G\|_{Q,\varphi_\varepsilon^{-1}} + \rho^{-\frac{1}{2}}\|h\|_{\Sigma_\varepsilon,\varphi_\varepsilon^{-1}\gamma^{-1}}\right)\|((\varepsilon\partial_t - \Delta + q)f, \partial_\nu f_{|\Sigma_{-\varepsilon,\omega}})\|_{L^2_{Q,\varphi_\varepsilon}\oplus L^2_{\Sigma_{-\varepsilon},\varphi_\varepsilon\gamma}},
\end{aligned}$$

where C is the constant in (4.3). By Hahn Banach's extension theorem, $\mathscr{S}$ is extended to a continuous linear form on $L^2_{Q,\varphi_\varepsilon} \oplus L^2_{\Sigma_{-\varepsilon},\varphi_\varepsilon\gamma}$. This extension is denoted again by $\mathscr{S}$. Additionally

$$\|\mathscr{S}\| \leq C\left(\rho^{-1}\|G\|_{Q,\varphi_\varepsilon^{-1}} + \rho^{-\frac{1}{2}}\|h\|_{\Sigma_\varepsilon,\varphi_\varepsilon^{-1}\gamma^{-1}}\right), \tag{4.4}$$

where $\|\mathscr{S}\|$ denotes the norm of $\mathscr{S}$ in $\left[L^2_{Q,\varphi_\varepsilon} \oplus L^2_{\Sigma_{-\varepsilon},\varphi_\varepsilon\gamma}\right]'$. Therefore, by Riesz's representation theorem, there exists $(z, g) \in L^2_{Q,\varphi_\varepsilon} \oplus L^2_{\Sigma_{-\varepsilon},\varphi_\varepsilon\gamma}$ so that, for any $f \in \mathcal{D}_\varepsilon$,

$$\mathscr{S}[((\varepsilon\partial_t - \Delta + q)f, \partial_\nu f_{|\Sigma_{-\varepsilon,\omega}})] = ((\varepsilon\partial_t - \Delta + q)f, z)_{L^2(Q)} + \langle \partial_\nu f, g\rangle_{L^2(\Sigma_{-\varepsilon,\omega})}.$$

In other words we proved that, for any $f \in \mathcal{D}_\varepsilon$,

$$((\varepsilon\partial_t - \Delta + q)f, z)_{L^2(Q)} + \langle \partial_\nu f, g\rangle_{L^2(\Sigma_{-\varepsilon,\omega})} = (f, G)_{L^2(Q)} - \langle \partial_\nu f, h\rangle_{L^2(\Sigma_{\varepsilon,\omega})}. \tag{4.5}$$

Taking $f \in C_0^\infty(Q)$, we get $(-\varepsilon\partial_t - \Delta + q)z = G$ in Q. Whence $z \in H_{-\varepsilon}$. In light of the trace theorem of Proposition 2.1, the other properties of z can be proved in a straightforward manner. $\square$

Proof of Theorem 4.1. We give only the existence of $u_{+,q}$. That of $u_{-,q}$ can be established following the same method and therefore we omit the proof in this case. In the sequel $\rho \geq \rho_0$. Recall that we seek $u_{+,q}$ in the form $u_{+,q} = \psi_-(\theta_+ + w_{+,q})$ which means that $w_{+,q}$ must be the solution of the IBVP

$$\begin{cases} (\partial_t - \Delta + q)(\psi_- w) = -(\partial_t - \Delta + q)(\psi_-\theta_+) \ \text{ in } Q, \\ w_{|\Omega_+} = 0, \\ w_{|\Sigma_{+,-\omega,\delta}} = -\theta_+. \end{cases} \tag{4.6}$$

The following identity is used in the sequel

$$-(\partial_t - \Delta + q)(\psi_-\theta_+) = -\psi_- e^{-i(x,t)\cdot\zeta}\left[(-i\tau + |\xi|^2 + q_1)\left(1 - e^{-\rho^{\frac{3}{4}}t}\right) - \rho^{\frac{3}{4}} e^{-\rho^{\frac{3}{4}}t}\right].$$

Pick $\varphi \in C_0^\infty(\mathbb{R}^n)$ so that $\mathrm{supp}(\varphi)\cap\Gamma \subset \{x \in \Gamma;\ \omega\cdot\nu(x) < -2\delta/3\}$ and $\varphi = 1$ on $\{x \in \Gamma;\ \omega\cdot\nu(x) < -\delta\} = \Gamma_{+,-\omega,\delta}$. Let

$$h = -\varphi\theta_+ \ \text{ on } \Sigma_{-,\omega} = \Sigma_{+,-\omega},$$
$$G = -\psi_- e^{-i(x,t)\cdot\zeta}\left[(-i\tau + |\xi|^2 + q_1)\left(1 - e^{-\rho^{\frac{3}{4}}t}\right) - \rho^{\frac{3}{4}} e^{-\rho^{\frac{3}{4}}t}\right].$$

From Lemma 4.1 (with $\varepsilon = -$), there exists $z \in H_+$ satisfying

$$\begin{cases} (\partial_t - \Delta + q)z = G \ \text{ in } Q, \\ z_{|\Omega_+} = 0, \\ z_{|\Sigma_{-,\omega}} = h. \end{cases}$$

As $\psi_+ = \psi_-^{-1}$, we see that $w_{+,q} = \psi_+ z$ satisfies (4.6).

We complete the proof by showing that inequality (4.2) holds for $w_{+,q}$. To do that, we firstly note that

$$\|G\|_{Q,\varphi_+} \leqslant C(|\tau| + |\xi|^2 + \rho^{\frac{3}{4}}) \leqslant C(\langle\zeta\rangle^2 + \rho^{\frac{3}{4}}),$$

for some constant $C > 0$ depending only on m and Q. Whence

$$\begin{aligned} \|w_{+,q}\|_{L^2(Q)} = \|z\|_{Q,\varphi_+} &\leqslant C\left(\rho^{-1}\|G\|_{Q,\varphi_-^{-1}} + \rho^{-\frac{1}{2}}\|h\|_{\Sigma_-,\varphi_-^{-1}\gamma^{-1}}\right) \\ &\leqslant C\left(\rho^{-1}\|G\|_{Q,\varphi_+} + \rho^{-\frac{1}{2}}\|h\|_{\Sigma_-,\varphi_+\gamma^{-1}}\right) \\ &\leqslant C\left(\rho^{-1}(\langle\zeta\rangle^2 + \rho^{\frac{3}{4}}) + \rho^{-\frac{1}{2}}\left\|\varphi\gamma^{-\frac{1}{2}}\right\|_{L^2(\Sigma_{-,\omega})}\right) \\ &\leqslant C(\rho^{-\frac{1}{4}} + \rho^{-1}\langle\zeta\rangle^2), \end{aligned}$$

as it is expected. □

5. Proof of Theorems 1.1 and 1.2

Fix $q, \widetilde{q} \in mB_{L^\infty(Q)}$ and set $p = (q - \widetilde{q})\chi_Q$, where χ_Q denotes the characteristic function of Q. For $\delta \in (0,1)$, let $\chi_{\pm,\omega,\delta} \in C_0^\infty(\mathbb{R}^n)$ satisfying

$$\mathrm{supp}(\chi_{\pm,\omega,\delta})\cap\Gamma \subset \Gamma_{-,\mp\omega,2\delta} \quad \text{and} \quad \chi_{\pm,\omega,\delta} = 1 \ \text{ on } \Gamma_{-,\mp\omega,\delta}.$$

As usual, the operator $\chi_{-,\omega,\delta}(\Lambda_q - \Lambda_{\widetilde{q}})\chi_{+,\omega,\delta}$ acts as follows

$$\chi_{-,\omega,\delta}(\Lambda_q - \Lambda_{\widetilde{q}})\chi_{+,\omega,\delta}(g) = \chi_{-,\omega,\delta}(\Lambda_q - \Lambda_{\widetilde{q}})(\chi_{+,\omega,\delta}g), \quad g \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma).$$

Recall that the Fourier transform $\widehat{p}$ of p is given by

$$\widehat{p}(\zeta) = (2\pi)^{-(n+1)/2}\int_{\mathbb{R}^{n+1}} p(x,t)e^{-i\zeta\cdot(x,t)}dxdt.$$

We start with the preliminary lemma

Lemma 5.1. *Let $\delta\in(0,1)$, $\omega\in\mathbb{S}^{n-1}$, $\zeta=(\xi,\tau)\in\mathbb{R}^n\times\mathbb{R}$ with $\xi\cdot\omega=0$ and ρ_0 be as in Theorem 3.1. Then*

$$|\widehat{p}(\zeta)|\leqslant C\left(\rho^{-\frac{1}{4}}+\rho^{-1}R^2+\|\chi_{-,\omega,\delta}(\Lambda_q-\Lambda_{\widetilde{q}})\chi_{+,\omega,\delta}\|\, e^{c\rho^2}\right),\ \ |\zeta|\leqslant R,\ \rho\geq\rho_0. \tag{5.1}$$

for some constants $c>0$ and $C>0$ depending only on m, δ and Q.

Here $\|\chi_{-,\omega,\delta}(\Lambda_q-\Lambda_{\widetilde{q}})\chi_{+,\omega,\delta}\|$ denotes the norm $\chi_{-,\omega,\delta}(\Lambda_q-\Lambda_{\widetilde{q}})\chi_{+,\omega,\delta}$ in $\mathscr{B}(H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma),H^{\frac{1}{2},\frac{1}{4}}(\Sigma))$.

Proof. Let $g=\tau_0u_{+,q}$ (resp. $h=u_{-,\widetilde{q}}$), with $u_{+,q}$ (resp. $u_{-,\widetilde{q}}$) as in Theorem 4.1. Then formula (2.11) yields

$$\left|\int_{\mathbb{R}^{n+1}}p(x,t)e^{-i\zeta\cdot(x,t)}dxdt\right|\leqslant\left|\int_Q Z(t,x)dxdt\right|+|\langle\chi_{-,\omega,\delta}(\Lambda_q-\Lambda_{\widetilde{q}})\chi_{+,\omega,\delta}g,h\rangle| \tag{5.2}$$

with

$$Z=e^{-i\zeta\cdot(x,t)}w_{-,\widetilde{q}}+w_{+,q}+w_{+,q}w_{-,\widetilde{q}}+e^{-\rho^{\frac{3}{4}}(T-t)}(e^{-i\zeta\cdot(x,t)}+w_{+,q})+e^{-\rho^{\frac{3}{4}}t}(1+w_{-,\widetilde{q}})+e^{-\rho^{\frac{3}{4}}T}.$$

In light of (4.2) and noting that

$$\int_0^T e^{-2\rho^{\frac{3}{4}}t}dt=\int_0^T e^{-2\rho^{\frac{3}{4}}(T-t)}dt\leqslant\int_0^{+\infty}e^{-2\rho^{\frac{3}{4}}t}dt=\frac{\rho^{-\frac{3}{4}}}{2},$$

we get by applying Cauchy-Schwarz's inequality

$$\left|\int_Q Z(t,x)dxdt\right|\leqslant C\rho^{-\frac{1}{4}}. \tag{5.3}$$

Since $u_{+,q}\in H_+$,

$$\|g\|_{H^{-\frac{1}{4},-\frac{1}{2}}(\Sigma)}\leqslant C\,\|u_{+,q}\|_{H_+}\leqslant C\,\|u_{+,q}\|_{L^2(Q)}\leqslant e^{c\rho^2} \tag{5.4}$$

with $c=T+\sup_{x\in\overline{\Omega}}|x|$ and the same estimate holds for $w_{-,\widetilde{q}}$. Finally a combination of (5.2), (5.3) and (5.4) entails (5.1). □

Proof of Theorem 1.1. In this proof c and C are generic constants that can depend only on m, Q and s. By Lemma 5.1,

$$|\widehat{p}(\zeta)|\leqslant C\left(\rho^{-\frac{1}{4}}+\rho^{-1}R^2+\|\Lambda_q-\Lambda_{\widetilde{q}}\|\, e^{c\rho^2}\right),\ \ |\zeta|\leqslant R,\ \rho\geq\rho_0. \tag{5.5}$$

On the other hand

$$\begin{aligned}\|p\|^2_{H^{-1}(\mathbb{R}^{n+1})}&=\int_{\mathbb{R}^{n+1}}(1+|\zeta|^2)^{-1}|\widehat{p}(\zeta)|^2d\zeta\\&=\int_{|\zeta|\leq R}(1+|\zeta|^2)^{-1}|\widehat{p}(\zeta)|^2d\zeta+\int_{|\zeta|\geq R}(1+|\zeta|^2)^{-1}|\widehat{p}(\zeta)|^2d\zeta\\&\leq|\{|\zeta|\leq R\}|\max_{|\zeta|\leq R}|\widehat{p}(\zeta)|^2+R^{-2}\|\widehat{p}\|^2_{L^2}\\&\leq|\{|\zeta|\leq R\}|\max_{|\zeta|\leq R}|\widehat{p}(\zeta)|^2+R^{-2}\|p\|^2_{L^2}\\&\leq|\{|\zeta|\leq R\}|\max_{|\zeta|\leq R}|\widehat{p}(\zeta)|^2+R^{-2}m^2|Q|.\end{aligned}$$

Whence, as a consequence of (5.5),

$$\|p\|^2_{H^{-1}(\mathbb{R}^{n+1})}\leqslant C\left(R^{-2}+\rho^{-\frac{1}{2}}R^{n+1}+\rho^{-2}R^{n+5}+R^{n+1}\gamma^2e^{c\rho^2}\right),\ \ R>0,\ \rho\geq\rho_0, \tag{5.6}$$

where we used the temporary notation $\gamma=\|\Lambda_q-\Lambda_{\widetilde{q}}\|$. In this inequality, we take $R=\rho^s$ in order to obtain, where $\alpha=1/2-s(n+1)(>0)$,

$$\|p\|^2_{H^{-1}(\mathbb{R}^{n+1})}\leq C\left(\rho^{-\alpha}+\gamma^2e^{c\rho^2}\right),\rho\geq\rho_1.$$

Here $\rho_1 \geq \rho_0$ is constant depending only on m and Q. A straightforward minimization argument with respect to ρ yields

$$\|p\|_{H^{-1}(\mathbb{R}^{n+1})} \leq C|\ln\gamma|^{-\alpha/4}, \tag{5.7}$$

if $\gamma \leq \gamma^*$, where γ^* is constant that can depend only on m, Q and s.

When $\gamma \geq \gamma^*$ we obtain, by using that $\|p\|_{H^{-1}(\mathbb{R}^{n+1})} \leq C\|p\|_{L^2(\mathbb{R}^{n+1})} \leq C|Q|^{1/2}m$,

$$\|p\|_{H^{-1}(\mathbb{R}^{n+1})} \leq C \leq C\frac{\gamma}{\gamma^*}. \tag{5.8}$$

We complete the proof by noting that (1.3) is obtained by combining (5.7) and (5.8). □

Before we proceed to the proof Theorem 1.2, we recall a result quantifying the unique continuation of a real-analytic function from a measurable set.

Theorem 5.1. ([2, Theorem 4]) *Assume that $H : 2\mathbb{B} \to \mathbb{C}$, where $\mathbb{B}$ the unit ball of $\mathbb{R}^{n+1}$, is real-analytic and satisfies*

$$|\partial^\alpha H(\kappa)| \leqslant K\frac{|\alpha|!}{\lambda^{|\alpha|}},\ \ \kappa \in 2\mathbb{B},\ \ \alpha \in \mathbb{N}^n,$$

for some $(K,\lambda) \in \mathbb{R}_+^* \times (0,1]$. Then for any measurable set $E \subset \mathbb{B}$ with positive Lebesgue measure, there exist two constants $M > 0$ and $\theta \in (0,1)$, depending only on λ and $|E|$, so that

$$\|H\|_{L^\infty(\mathbb{B})} \leqslant MK^{1-\theta}\left(\frac{1}{|E|}\int_E |H(\kappa)|d\kappa\right)^\theta.$$

Proof of Theorem 1.2. Let $\delta > 0$ be chosen in such a way that, for any $\omega \in O_\delta = \{\eta \in \mathbb{S}^{n-1};\ |\eta - \omega_0| \leqslant \delta\}$,

$$\Gamma_{-,-\omega,3\delta} \subset \mathcal{U}_- \ \text{ and } \ \Gamma_{-,\omega,3\delta} \subset \mathcal{U}_+.$$

In the rest of this proof, C and c denote generic constants that can depend only on Ω, Q, m and $\mathcal{U}_\pm$. Define

$$E_1 := \bigcup_{\omega\in O_\delta}\{(\xi,\tau) \in \mathbb{R}^n \times \mathbb{R};\ \xi\cdot\omega = 0\}, \quad E := \{\zeta = (\xi,\tau) \in E_1;\ |\zeta| < 1\}.$$

In light of estimate (5.1), for all $\zeta = (\xi,\tau) \in \{\eta \in E_1;\ |\eta| \leqslant R\}$, there exists $\omega \in O_\delta$ satisfying $\xi\cdot\omega = 0$ such that

$$|\widehat{p}(\zeta)| \leqslant C\left(\rho^{-\frac{1}{4}} + \rho^{-1}R^2 + \|\chi_{-,\omega,\delta}(\Lambda_q - \Lambda_{\widetilde{q}})\chi_{+,\omega,\delta}\|\, e^{c\rho^2}\right),\ \ \rho \geq \rho_0.$$

But, for any $\omega \in O_\delta$, $\mathrm{supp}(\chi_{\pm,\omega,\delta}) \subset \mathcal{U}_\pm$. Whence

$$\|\chi_{-,\omega,\delta}(\Lambda_q - \Lambda_{\widetilde{q}})\chi_{+,\omega,\delta}\| \leqslant C\left\|\widehat{\Lambda}_q - \widehat{\Lambda}_{\widetilde{q}}\right\|.$$

Therefore

$$|\widehat{p}(\zeta)| \leqslant C\left(\rho^{-\frac{1}{4}} + \rho^{-1}R^2 + \left\|\widehat{\Lambda}_q - \widehat{\Lambda}_{\widetilde{q}}\right\| e^{c\rho^2}\right),\ \ \zeta \in \{\eta \in E_1;\ |\eta| \leqslant R\},\ \rho \geq \rho_0. \tag{5.9}$$

Let

$$H(\zeta) = \widehat{p}(R\zeta).$$

We repeat the same argument as in the proof of [34, Theorem 1] in order to get

$$|\partial^\alpha H(\zeta)| \leqslant C\frac{e^{2R}|\alpha|!}{\lambda^{|\alpha|}},\ \ \zeta \in 2\mathbb{B},\ \ \alpha \in \mathbb{N}^{n+1},$$

where $\lambda = [1 + \max(T, \mathrm{Diam}(\Omega))]^{-1} \in (0,1)$. An application of Theorem 5.1 with $K = e^{2R}$ yields

$$|\widehat{p}(R\zeta)| = |H(\zeta)| \leqslant \|H\|_{L^\infty(\mathbb{B})} \leqslant Ce^{2R(1-\theta)}\|H\|^\theta_{L^\infty(E)},\ \ |\zeta| < 1,$$

for some $0 < \theta < 1$ depending on Q and $\mathcal{U}_\pm$. But, from (5.9) we deduce

$$|\widehat{p}(R\zeta)|^2 = |H(\zeta)|^2 \leqslant C\left(\rho^{-\frac{1}{2}} + \rho^{-2}R^4 + \left\|\widehat{\Lambda}_q - \widehat{\Lambda}_{\widetilde{q}}\right\|^2 e^{c\rho^2}\right),\ \ \zeta \in E,\ \rho \geq \rho_0.$$

Consequently, fixing $\sigma = 4(1-\theta)$, we find

$$|\widehat{p}(\zeta)|^2 \leqslant Ce^{\sigma R}\left(\rho^{-\frac{1}{2}} + \rho^{-2}R^4 + \left\|\widehat{\Lambda}_q - \widehat{\Lambda}_{\widetilde{q}}\right\|^2 e^{c\rho^2}\right)^{\theta},\ |\zeta| < R,\ \rho \geq \rho_0. \tag{5.10}$$

We proceed as in the proof of Theorem 1.1 in order to obtain

$$\|p\|^{\frac{2}{\theta}}_{H^{-1}(\mathbb{R}^{n+1})} \leqslant C\left(R^{-2} + R^{n+1}e^{\sigma R}\left(\rho^{-\frac{1}{2}} + \rho^{-2}R^4 + \left\|\widehat{\Lambda}_q - \widehat{\Lambda}_{\widetilde{q}}\right\|^2 e^{c\rho^2}\right)^{\theta}\right)^{\frac{1}{\theta}},\ \ R > 0,\ \rho \geq \rho_0.$$

As $\mu \in (0,\infty) \to \mu^{\frac{1}{\theta}}$ is convex, we derive

$$\|p\|^{\frac{2}{\theta}}_{H^{-1}(\mathbb{R}^{n+1})} \leq C\left(R^{-\frac{2}{\theta}} + \left[R^{n+1}e^{\sigma R}\right]^{\frac{1}{\theta}}\left(\rho^{-\frac{1}{2}} + \rho^{-2}R^4 + \left\|\widehat{\Lambda}_q - \widehat{\Lambda}_{\widetilde{q}}\right\|^2 e^{c\rho^2}\right)\right),\ \ R > 0,\ \rho \geq \rho_0.$$

Hence, there exists $\beta > 0$ depending only on n and θ so that

$$\|p\|^{\frac{2}{\theta}}_{H^{-1}(\mathbb{R}^{n+1})} \leq C\left(R^{-\frac{2}{\theta}} + e^{\beta R}\rho^{-\frac{1}{2}} + e^{\beta R}\left\|\widehat{\Lambda}_q - \widehat{\Lambda}_{\widetilde{q}}\right\|^2 e^{c\rho^2}\right),\ \ R > 0,\ \rho \geq \rho_0.$$

Thus, there exists $R_0 > 0$ so that choosing ρ satisfying $e^{\beta R}\rho^{-\frac{1}{2}} = R^{-\frac{2}{\theta}}$ we find

$$\|p\|^{\frac{2}{\theta}}_{H^{-1}(\mathbb{R}^{n+1})} \leq C\left(R^{-\frac{2}{\theta}} + \left\|\widehat{\Lambda}_q - \widehat{\Lambda}_{\widetilde{q}}\right\|^2 e^{e^R}\right),\ \ R \geq R_0.$$

The proof is completed similarly to that of Theorem 1.1 by using a minimization argument. □

6. Determining the nonlinear term in a semi-linear IBVP from the DN map

The objective in the actual section is the derivation of a stability estimate of the problem of determining the nonlinear term in a semi-linear parabolic IBVP from the corresponding "linearized" DtN map. We will give the precise definition of the "linearized" DtN map later in the text. The results of this section are obtained as a consequence of Theorem 1.1.

The linearization procedure we use require existence, uniqueness and a priori estimate of solutions of IBVP's under consideration. We preferred to work in the Hölder space setting for which we have a precise literature devoted to these aspects of solutions. However we are convinced that the same analysis can be achieved in the Sobolev space setting. But in that case this analysis seems to be more delicate.

In this section Ω is of class $C^{2+\alpha}$ for some $0 < \alpha < 1$. The parabolic boundary of Q is denoted by Σ_p. That is $\Sigma_p = \Sigma \cup \Omega_+$.

Consider the semilinear IBVP for the heat equation

$$\begin{cases} (\partial_t - \Delta)u + a(x,t,u) = 0 & \text{in } Q, \\ u = g & \text{on } \Sigma_p. \end{cases} \tag{6.1}$$

We introduce some notations. We denote by $\mathscr{A}_0$ the set of functions from $C^1(\overline{Q} \times \mathbb{R})$ satisfying one of the following condition

(i) There exist two non negative constants c_0 and c_1 so that

$$ua(x,t,u) \geq -c_0u^2 - c_1,\ \ (x,t,u) \in \overline{Q} \times \mathbb{R}. \tag{6.2}$$

(ii) There exist a non negative constant c_2 and a non decreasing positive function of $\tau \geq 0$ satisfying

$$\int_0^\infty \frac{d\tau}{\Phi(\tau)} = \infty$$

so that

$$ua(x,t,u) \geq -|u|\Phi(|u|) - c_2,\ \ (x,t,u) \in \overline{Q} \times \mathbb{R}. \tag{6.3}$$

Set $X = C^{2+\alpha,1+\alpha/2}(\overline{Q})$ and let $X_0 = \{g = G_{|\overline{\Sigma}_p};\ \text{for some } G \in X\}$. If $\|\cdot\|_X$ denotes the natural norm on X we equip X_0 with the quotient norm

$$\|g\|_{X_0} = \inf\{\|G\|_X;\ G_{|\overline{\Sigma}_p} = g\}.$$

By [Theorem 6.1, page 452, LSU], for any $a \in \mathscr{A}_0$ and $g \in X_0$, the IBVP (6.1) has a unique solution $u_{a,g} \in X$. Additionally, according to [Theorem 2.9, page 23, LSU], there exists a constant C that can depend only on Q, $\mathscr{A}_0$ and $\max\limits_{\overline{\Sigma}_p}|g|$ such that

$$\max_{\overline{Q}} |u_{a,g}| \le C. \tag{6.4}$$

A quick inspection of [inequalities (2.31) and (2.34), page 23, LSU] shows that

$$\max_{\overline{\Sigma}_p}|g| \to C = C(\max_{\overline{\Sigma}_p}|g|)$$

is non decreasing.

Define the parabolic DtN map N_a associated to $a \in \mathscr{A}_0$ by

$$N_a : g \in X_0 \longrightarrow \partial_\nu u_{a,g} \in Y = C^{1+\alpha,(1+\alpha)/2}(\overline{\Sigma}).$$

Note that, contrary to the preceding case, actually the DtN map N_a is no longer linear. The linearization procedure consists then in computing the Fréchet derivative of N_a.

Let $\mathscr{A}$ be the subset of $\mathscr{A}_0$ of those functions a satisfying $\partial_u a \in C^2(\overline{Q} \times \mathbb{R})$. For $a \in \mathscr{A}$ and $h \in X_0$, consider the IBVP

$$\begin{cases} (\partial_t - \Delta)v + \partial_u a(x,t,u_{a,g}(x,t))v = 0 & \text{in } Q, \\ v = h & \text{on } \Sigma_p. \end{cases}$$

In light of [Theorem 5.4, page 322, LSU] the IBVP has a unique solution $v = v_{a,g,h} \in X$ satisfying

$$\|v_{a,g,h}\|_X \le c\|h\|_{X_0}$$

for some constant c depending only on Q, a and g. In particular $h \in X_0 \to v_{a,g,h} \in X$ defines a bounded operator.

Proposition 6.1. *For each $a \in \mathscr{A}$, N_a is continuously Fréchet differentiable and*

$$N_a'(g)(h) = \partial_\nu v_{a,g,h} \in Y,\ \ g,h \in X_0.$$

Proof. Let $a \in \mathscr{A}$. As $u \in X \to \partial_\nu u \in Y$ is a bounded linear operator, it is enough to prove that $M_a : g \in X_0 \to u_{a,g} \in X$ is continuously differentiable and $M_a'(g)(h) = v_{a,g,h}$, $g,h \in X_0$. To do that, we define $w \in X$ by

$$w = u_{a,g+h} - u_g - v_{a,g,h}$$

and set

$$\begin{aligned} p(x,t) &= \partial_u a(x,t,u_{a,g}(x,t)), \\ q(x,t) &= \int_0^1 (1-\tau)\partial_u^2 a(x,t,u_{a,g} + \tau(u_{a,g+h} - u_{a,g}))d\tau. \end{aligned}$$

According to Taylor's formula

$$\begin{aligned} a(x,t,u_{a,g+h}(x,t)) - a(x,t,u_{a,g}(x,t)) &= p(x,t)(u_{a,g+h}(x,t) - u_{a,g}(x,t)) \\ &\quad + q(x,t)(u_{a,g+h}(x,t) - u_{a,g}(x,t))^2. \end{aligned}$$

Consequently

$$\begin{aligned} a(x,t,u_{a,g+h}(x,t)) - a(x,t,u_{a,g}(x,t)) - p(x,t)v_{a,g,h}(x,t) &= p(x,t)w(x,t) \\ &\quad + q(x,t)(u_{a,g+h}(x,t) - u_{a,g}(x,t))^2. \end{aligned}$$

Moreover, it is straightforward to check that w is the solution of the IBVP

$$\begin{cases} (\partial_t - \Delta + p)w = -q(u_{a,g+h} - u_{a,g})^2 & \text{in } Q, \\ w = 0 & \text{on } \Sigma_p. \end{cases}$$

Set $Z = C^{\alpha,\alpha/2}(\overline{Q})$. From inequality (6.4) and the comment following it,

$$\max_{\overline{Q}} |u_{a,g+h}| \le c, \ \text{ for any } h \in B_{X_0}.$$

Here and in the rest of this proof, c is a generic constant can depend only on Q, a and g. Whence

$$\|p\|_Z,\ \|q\|_Z \le c, \ \text{ for any } h \in B_{X_0},$$

Therefore, again by [Theorem 5.4, page 322, LSU], it holds

$$\|w\|_X \le c\|u_{a,g+h} - u_{a,g}\|_Z^2, \ \text{ for any } h \in B_{X_0}. \tag{6.5}$$

On the other hand $z = u_{a,g+h} - u_{a,g}$ is the solution of the IBVP

$$\begin{cases} (\partial_t - \Delta + r(x,t))z = 0 & \text{in } Q, \\ z = h & \text{on } \Sigma_p, \end{cases}$$

with

$$r(x,t) = \int_0^1 \partial_u a(x,t,u_{a,g} + \tau(u_{a,g+h} - u_{a,g}))d\tau.$$

Proceeding as above, we get

$$\|z\|_X \le c\|h\|_{X_0}.$$

This estimate, combined with (6.5), yields

$$\|w\|_X \le c\|h\|_{X_0}^2.$$

That is we proved that M_a is differentiable at g and $M_a'(g)(h) = v_{a,g,h}$, $h \in X_0$.

It remains to establish the continuity of $g \in X_0 \to M_a'(g) \in \mathscr{B}(X_0, X)$. To this end, let $g, k, h \in X_0$ and set

$$\varphi = v_{a,g+k,h} - v_{a,g,h}.$$

We see that φ is the unique solution of the IBVP

$$\begin{cases} [\partial_t - \Delta + \partial_u a(x,t,u_{a,g+k}(x,t))]\,\varphi = \alpha(x,t)(u_{a,g} - u_{a,g+k})v_{a,g,h} & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma_p, \end{cases}$$

where

$$\alpha(x,t) = \int_0^1 \partial_u^2 a(x,t,u_{a,g}(x,t) + \tau(u_{a,g+k}(x,t) - u_{a,g}(x,t)))d\tau.$$

Assume that $k, h \in B_{X_0}$. By proceeding one more time as above we get

$$\|\varphi\|_X \le c\|u_{a,g+k} - u_{a,g}\|_Z \|v_{a,g,h}\|_Z \le c\|u_{a,g+k} - u_{a,g}\|_Z.$$

Whence

$$\|M_a'(g+k) - M_a'(g)\|_{\mathscr{B}(X_0,X)} \le c\|M_a(g+k) - M_a(g)\|_X,$$

which leads immediately to the continuity of M_a' since M_a is continuous. □

In order to handle the inverse problem corresponding to the semi-linear IBVP (6.1) we need to extend the operator Λ_q by varying also the initial condition. To do that we start by considering the IBVP

$$\begin{cases} (\partial_t - \Delta + q(t,x))u = 0 \ \text{ in } Q, \\ u_{|\Omega_+} = u_0, \\ u_{|\Sigma} = g. \end{cases} \tag{6.6}$$

Let $X_+ = r_+ H_+ \subset H^{-1}(\Omega)$ that we equip with its natural quotient norm

$$\|u_0\|_{X_+} = \inf\{\|u\|_{H_+};\ r_+ u = u_0\}.$$

Let $(u_0,g)\in X_+\oplus H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ and pick $v\in H_+$ so that $r_+v=u_0$. Formally, if u is a solution of the IBVP (6.7) then $w=u-v$ is the solution of the IBVP

$$\begin{cases} (\partial_t-\Delta+q(t,x))w=f \ \text{ in } Q,\\ w_{|\Omega_+}(0,\cdot)=0,\\ w_{|\Sigma}=h=g-\tau_0 v\in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma).\end{cases}\tag{6.7}$$

Here

$$f=-\partial_t v+\Delta v-q(t,x)v\in L^2(Q).$$

Fix $q\in mB_{L^\infty(Q)}$. When $g=0$ (resp. $f=0$) the IBVP problem (6.7) has a unique solution $w_{q,f}^1\in H^{2,1}(Q)$ [14, Theorem 1.43, page 27] (resp. $w_{q,h}^2\in H_+$ by Proposition 2.3) and

$$\|w_{q,f}^1\|_{H^{2,1}(Q)}\le C_0\|f\|_{L^2(Q)}\le C_1\|v\|_{H_+},$$
$$\|w_{q,h}^2\|_{H_+}\le C_2\|g-\tau_0 v\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}\le C_3\left(\|g\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}+\|v\|_{H_+}\right),$$

for some constants C_i depending only on Q and m. Whence, $u_{q,u_0,g}=v+w_{q,f}^1+w_{q,h}^2\in H_+$ is the unique solution of the IBVP (6.7) and there exists a constant $C>0$ that can depend only on Q and m so that

$$\|u_{q,u_0,g}\|_{H_+}\le C\left(\|g\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}+\|v\|_{H_+}\right).$$

Since $v\in H_+$ is chosen arbitrary so that $r_+v=u_0$, we derive

$$\|u_{q,u_0,g}\|_{H_+}\le C\left(\|g\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}+\|u_0\|_{X_+}\right).\tag{6.8}$$

Therefore, according to the trace theorem in Proposition 2.1, the extended parabolic DtN map

$$\Lambda_q^e : X_+\oplus H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)\to H^{-\frac{3}{2},-\frac{3}{4}}(\Sigma)$$
$$(u_0,g)\mapsto \tau_1 u_{q,u_0,g}$$

defines a bounded operator.

We need a variant of Theorem 1.2. To this end, we recall that from [37, Lemma 12.3, page 73] we have the following interpolation inequality, where c_0 is a constant depending only on Q,

$$\|u\|_{L^2(Q)}\le c_0\|u\|_{H^1(Q)}^{1/2}\|u\|_{H^{-1}(Q)}^{1/2},\ \ u\in H^1(Q).\tag{6.9}$$

On the other hand

$$\|u\|_{C(\overline{Q})}\le c_1\|u\|_{C^1(\overline{Q})}^{\frac{n+1}{n+3}}\|u\|_{L^2(Q)}^{\frac{2}{n+3}},\ \ u\in C^1(\overline{Q}),\tag{6.10}$$

where c_1 is a constant depending only on Q.

This interpolation inequality is more or less known but, for sake of completeness, we provide its proof in Lemma B.1 of Appendix B.

We get by combining these two interpolation inequalities the following one

$$\|u\|_{C(\overline{Q})}\le c\|u\|_{C^1(\overline{Q})}^{\frac{n+2}{n+3}}\|u\|_{H^{-1}(Q)}^{\frac{1}{n+3}},\ \ u\in C^1(\overline{Q}),\tag{6.11}$$

for some constant c depending only on Q.

If $\frac{1}{2(n+3)}<s<\frac{1}{2(n+1)}$,

$$\Theta_s(\rho)=|\ln\rho|^{-\frac{1-2s(n+1)}{n+3}}+\rho,\ \ \rho>0,\tag{6.12}$$

extended by continuity at $\rho=0$ by setting $\Theta_s(0)=0$.

Inspecting the proof of Theorem 1.1, using the interpolation inequality (6.11) and that

$$\|\Lambda_q-\Lambda_{\tilde q}\|\le\left\|\Lambda_q^e-\Lambda_{\tilde q}^e\right\|,$$

we get

Theorem 6.1. *Fix $m>0$ and $\frac{1}{2(n+3)}<s<\frac{1}{2(n+1)}$. There exists a constant $C>0$, that can depend only on m, Q and s, so that for any $q,\widetilde{q}\in mB_{C^1(\overline{Q})}$,*

$$\|q-\widetilde{q}\|_{C(\overline{Q})}\leqslant C\Theta_s\left(\left\|\Lambda_q^e-\Lambda_{\widetilde{q}}^e\right\|\right).$$

Here $\left\|\Lambda_q^e-\Lambda_{\widetilde{q}}^e\right\|$ stands for the norm of $\Lambda_q^e-\Lambda_{\widetilde{q}}^e$ in $\mathscr{B}(X_+\oplus H^{-\frac12,-\frac14}(\Sigma);H^{\frac12,\frac14}(\Sigma))$.

Fix $\lambda>0$. From (6.4) and the remark following it, there exists a constant $c_\lambda>0$ so that

$$\max_{\overline{Q}}|u_{a,g}|\le c_\lambda,\ \ a\in\mathscr{A}_0,\ \max_{\overline{\Sigma}_p}|g|\le\lambda.$$

For fixed $\delta>0$, consider

$$\widehat{\mathscr{A}}=\{a=a(x,u)\in\mathscr{A};\ \|\partial_u a\|_{C(\overline{\Omega}\times[-c_\lambda,c_\lambda])}\le\delta\}.$$

To $a\in\widehat{\mathscr{A}}$ and $g\in X_0$ we associate

$$p_{a,g}(x,t)=\partial_u a(x,u_{a,g}(x,t)),\ \ (x,t)\in\overline{Q}.$$

It is straightforward to check that

$$N_a'(g)=\Lambda^e_{p_{a,g}|X_0}.$$

From now on $N_a'(g)-N_{\widetilde{a}}'(g)$ is considered as a bounded operator from X_0 endowed with norm of $X_+\oplus H^{-\frac12,-\frac14}(\Sigma)$ into $H^{\frac12,\frac14}(\Sigma)$.

Since $\|p_{a,g}\|_{L^\infty(Q)}\le\delta$ for any $a\in\widehat{\mathscr{A}}$ and $g\in X_0$ so that $\max_{\overline{\Sigma}_p}|g|\le\lambda$, we get as a consequence of Proposition 2.3

$$\sup\{\|N_a'(g)-N_{\widetilde{a}}'(g)\|;\ a\in\widehat{\mathscr{A}},\ g\in X_0\text{ and }\max_{\overline{\Sigma}_p}|g|\le\lambda\}<\infty.$$

Bearing in mind that $C^\infty(\overline{Q})$ is dense in H_+, we derive that X_0 is dense $X_+\oplus H^{-\frac12,-\frac14}(\Sigma)$. Thus,

$$\|N_a'(g)-N_{\widetilde{a}}'(g)\|=\|\Lambda^e_{p_{a,g}}-\Lambda^e_{p_{\widetilde{a},g}}\|. \tag{6.13}$$

Pick $a_0\in C^1(\overline{\Omega})$ and set

$$\widehat{\mathscr{A}_0}=\{a\in\widehat{\mathscr{A}};\ a(\cdot,0)=a_0\}.$$

We note that when $g\equiv s$, $|s|\le\lambda$, we have

$$p_{a,g}(x,0)=\partial_u a(x,u_{a,g}(x,0))=\partial_u a(x,s),\ \ x\in\overline{\Omega}.$$

In light of this identity and (6.13) we obtain as a consequence of Theorem 6.1

Theorem 6.2. *Fix $\lambda>0$ and $\frac{1}{2(n+3)}<s<\frac{1}{2(n+1)}$. There exists a constant $C>0$, that can depend only on λ, s, Q and $\widehat{\mathscr{A}_0}$, so that for any $a,\widetilde{a}\in\widehat{\mathscr{A}_0}$,*

$$\|a-\widetilde{a}\|_{C(\overline{\Omega}\times[-\lambda,\lambda])}\leqslant C\Theta_s\left(\sup_{g\in X_{0,\lambda}}\|N_a'(g)-N_{\widetilde{a}}'(g)\|\right).$$

Here $X_{0,\lambda}=\{g\in X_0;\ \max_{\overline{\Sigma}_p}|g|\le\lambda\}$ and $\|N_a'(g)-N_{\widetilde{a}}'(g)\|$ stands for the norm of $N_a'(g)-N_{\widetilde{a}}'(g)$ in $\mathscr{B}(X_+\oplus H^{-\frac12,-\frac14}(\Sigma);H^{\frac12,\frac14}(\Sigma))$.

We now turn our attention to the special case $a=a(u)$ for which we are going to show that we have a stability estimate with less data than in the case $a=a(x,u)$.

Define

$$\mathcal{A}=\{a\in C^3(\mathbb{R});\ a(0)=0\text{ and }a\text{ is positive increasing}\}.$$

It is straightforward to check that $\mathcal{A}\subset\mathscr{A}$. Let

$$Y_0=C^{2+\alpha,1+\alpha/2}_{,0}(\overline{\Sigma})=\{g\in C^{2+\alpha,1+\alpha/2}(\overline{\Sigma});\ g(\cdot,0)=0\}$$

that we identify to the subset of X_0 given by $\{g \in X_0;\ g_{|\Omega_+} = 0\}$. We denote again the solution of (6.1) by $u_{a,g}$ when $a \in \mathcal{A}$ and $g \in Y_0$. In that case N_a is considered as a map from Y_0 into Y. Fix $g \in Y_0$ and let $a \in \mathcal{A}$. Since $a(0) = 0$, setting $q(t,x) = \int_0^1 a'(\tau u(x,t))d\tau$, we easily see that $u_{a,g}$ is also the solution of the IBVP

$$\begin{cases} (\partial_t - \Delta + q(x,t))u = 0 & \text{in } Q, \\ u = g & \text{on } \Sigma_p. \end{cases}$$

But $q \geq 0$. Whence

$$\min_{\overline{Q}} u = -\max_{\overline{\Sigma}} g^- = a(g), \quad \max_{\overline{Q}} u = \max_{\overline{\Sigma}} g^+ = b(g)$$

according to the weak maximum principle (see for instance [39, Theorem 4.25, page 121]). In particular

$$u(\overline{Q}) = I_g = [a(g), b(g)].$$

We derive by mimicking the analysis before Theorem 6.3

Theorem 6.3. *Fix $g \in Y_0$ non constant and $\frac{1}{2(n+3)} < s < \frac{1}{2(n+1)}$. There exists a constant $C > 0$, that can depend only on g, s, Q and $\mathcal{A}$, so that for any $a, \widetilde{a} \in \mathcal{A}$,*

$$\|a - \widetilde{a}\|_{C(I_g)} \leqslant C\Theta_s\left(\|N'_a(g) - N'_{\widetilde{a}}(g)\|\right).$$

Here $\|N'_a(g) - N'_{\widetilde{a}}(g)\|$ denotes the norm of $N'_a(g) - N'_{\widetilde{a}}(g)$ in $\mathscr{B}(H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma); H^{\frac{1}{2},\frac{1}{4}}(\Sigma))$.

Remark 6.1. Other stability results can be obtained in a similar manner to that we used in the present section. We just mention one of them. To this end, let $\widehat{\mathscr{A}_0}$ be defined as before with the only difference that we actually permit to functions of $\widehat{\mathscr{A}_0}$ to depend also on the time variable t.

Let $a, \widetilde{a} \in \widehat{\mathscr{A}_0}$ and pick $(x_0, t_0, u_0) \in \Gamma \times (0,T) \times [-\lambda, \lambda]$ so that

$$|(a - \widetilde{a})(x_0, t_0, u_0)| = \frac{1}{2}\|a - \widetilde{a}\|_{C(\Gamma \times [0,T] \times [-\lambda,\lambda])}. \tag{6.14}$$

Let $\varepsilon = \min(t_0, T - t_0)$ and $g \in X_{0,\lambda}$ so that $g = s$ on $\Gamma \times [\varepsilon, T - \varepsilon]$ for some $|s| \leq \lambda$. We proceed as in the proof of Theorem 6.3 in order to derive

$$\|a - \widetilde{a}\|_{C(\Gamma \times [\varepsilon, T-\varepsilon] \times [-\lambda,\lambda])} \leq C\Theta_s\left(\sup_{g \in X_{0,\lambda}} \|N'_a(g) - N'_{\widetilde{a}}(g)\|\right),$$

where the constant C depends only on λ, s, Q and $\widehat{\mathscr{A}_0}$.

In light of (6.14) this estimate yields

$$\|a - \widetilde{a}\|_{C(\Gamma \times [0,T] \times [-\lambda,\lambda])} \leq C\Theta_s\left(\sup_{g \in X_{0,\lambda}} \|N'_a(g) - N'_{\widetilde{a}}(g)\|\right).$$

Appendix A

Lemma A.1. *$H^s(0,T; L^2(\Gamma)) = H^s_0(0,T; L^2(\Gamma))$ for any $0 \leq s \leq 1/2$.*

Proof. Let $\mathcal{D}_0$ (resp. $\mathcal{D}_T$) be the subspace of $C_0^\infty(\mathbb{R}, L^2(\Gamma))$ of functions vanishing in a neighborhood of $t = 0$ (resp. $t = T$). By [37, Lemma 11.1, page 55] both $\mathcal{D}_0$ and $\mathcal{D}_T$ are dense in $H^{1/2}(\mathbb{R}, L^2(\Gamma))$.

Fix $u \in \mathcal{D}_0$ and let $\varepsilon > 0$ so that $u = 0$ in $(-2\varepsilon, 2\varepsilon) \times \Gamma$. As $\mathcal{D}_T$ is dense in $H^{1/2}(\mathbb{R}, L^2(\Gamma))$, there exists a sequence (u_n) in $\mathcal{D}_T$ that converges to u in $H^{1/2}(\mathbb{R}, L^2(\Gamma))$. Pick $\psi \in C_0^\infty(-2\varepsilon, 2\varepsilon)$ satisfying $\psi = 1$ on $(-\varepsilon, \varepsilon)$.

We get by taking into account that $\psi u_n = \psi(u_n - u)$

$$\|\psi u_n\|_{H^{\frac{1}{2}}(\mathbb{R}; L^2(\Gamma))} = \|\psi(u_n - u)\|_{H^{\frac{1}{2}}(\mathbb{R}; L^2(\Gamma))} \leqslant C\|u_n - u\|_{H^{\frac{1}{2}}(\mathbb{R}; L^2(\Gamma))},$$

for some constant depending only on ψ. Whence $((1-\psi)u_n)$ is a sequence in $\mathcal{D}_0 \cap \mathcal{D}_T$ converging to u in $H^{1/2}(\mathbb{R}, L^2(\Gamma))$.

On the other hand, we know that $H^{1/2}(0,T;L^2(\Gamma))$ can be seen as the (quotient) space of the restriction to $(0,T)$ of functions from $H^{1/2}(\mathbb{R},L^2(\Gamma))$. Therefore any function from $H^{1/2}(0,T;L^2(\Gamma))$ can be approximated, with respect to the norm of $H^{1/2}(0,T;L^2(\Gamma))$, by a sequence of functions from $\mathcal{D}_0\cap\mathcal{D}_T$, that is to say by a sequence from $C_0^\infty((0,T);L^2(\Gamma))$.

The proof follows by noting that $H^{\frac12}(0,T;L^2(\Gamma))$ is continuously and densely embedded in $H^s(0,T;L^2(\Gamma))$, $0\le s\le 1/2$. □

Appendix B

Lemma B.1. *Let D be a bounded domain of $\mathbb{R}^d$, $d\ge 2$, of class $C^{0,\alpha}$ with $0<\alpha\le 1$. There exists a constant $C>0$ depending only on D and α so that, for any $u\in C^{0,\alpha}(\overline{D})$,*

$$\|u\|_{C(\overline{D})}\le C\|u\|_{C^{0,\alpha}(\overline{D})}^{\frac{d}{d+2\alpha}}\|u\|_{L^2(D)}^{\frac{2\alpha}{d+2\alpha}}.$$

Proof. Let $u\in C^{0,\alpha}(\overline{D})$. From [26, Lemma 6.37, page 136. In fact this lemma is stated with $C^{k,\alpha}$-regularity, $k\ge 1$, but a careful inspection of the proof shows that this lemma can be extended to the case of $C^{0,\alpha}$-regularity] and its proof, there exists $v\in C^{0,\alpha}(\mathbb{R}^d)$ with compact support so that $v=u$ in $\overline{D}$ and

$$\|v\|_{C^{0,\alpha}(\mathbb{R}^d)}\le\kappa\|u\|_{C^{0,\alpha}(\overline{D})},\quad \|v\|_{L^2(\mathbb{R}^d)}\le\kappa\|u\|_{L^2(D)}, \tag{B.1}$$

where the constant κ depends only on D.

For $x\in\overline{D}$ and $r>0$, we get in a straightforward manner

$$\begin{aligned}|u(x)||B(x,r)| &\le \int_{B(x,r)}|v(x)-v(y)|dy+\int_{B(x,r)}|v(y)|dy\\ &\le \|v\|_{C^{0,\alpha}(\mathbb{R}^d)}\int_{B(x,r)}|x-y|^\alpha dy+|B(x,r)|^{1/2}\|v\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

This estimate combined with (B.1) yields

$$|u(x)|\le C\left(r^\alpha\|u\|_{C^{0,\alpha}(\overline{D})}+r^{-d/2}\|u\|_{L^2(D)}\right), \tag{B.2}$$

where the constant C depends only on D and α. The expected inequality follows by taking r in (B.2) so that $r^\alpha\|u\|_{C^{0,\alpha}(\overline{D})}=r^{-d/2}\|u\|_{L^2(D)}$. □

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